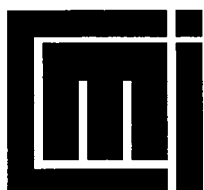# Evolutionary Game Theory and the Battle of the Sexes

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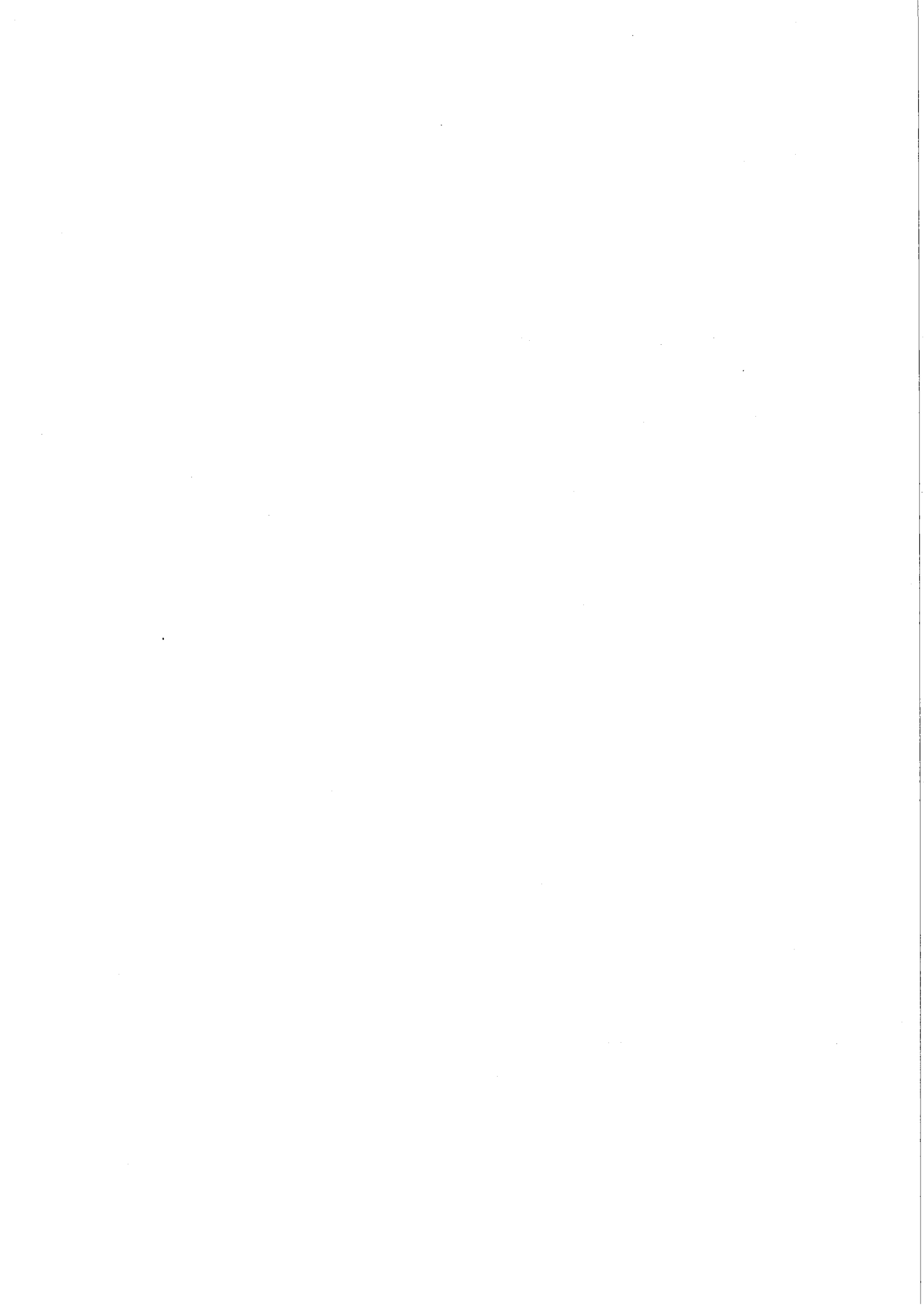Development Studies and Human Rights

Evolutionary Game Theory and the Battle of the Sexes

Turid Bøe

Summary:

This paper presents an outline of different approaches within evolutionary game theory and sees these approaches in relation to the problem of choosing among multiple equilibria in normal form games. More specifically the paper extends the analysis of Mailath, Kandori and Rob (1993) addressing 2x2 player symmetric games to cover 2x2 player asymmetric games. Utilising an evolutionary model with a finite and equal number of players within two different player populations and allowing for mutations perpetuating the system way from its deterministic evolution, we show that for the asymmetric "battle of the sexes" game the long run equilibrium (for large populations) chosen satisfies the Harsany and Selten (1988) criterion for risk dominance. In games with player-specific risk, that is in games where the players face identical risks regardless of the strategy chosen, but where the players from the different population face different risks, the population size N does not influence the equilibrium chosen. In such games the long run equilibrium is the equilibrium with the largest surplus product.

Indexing terms:
Game theory
Sex roles
Economic equilibrium
Dynamic processes

Contents
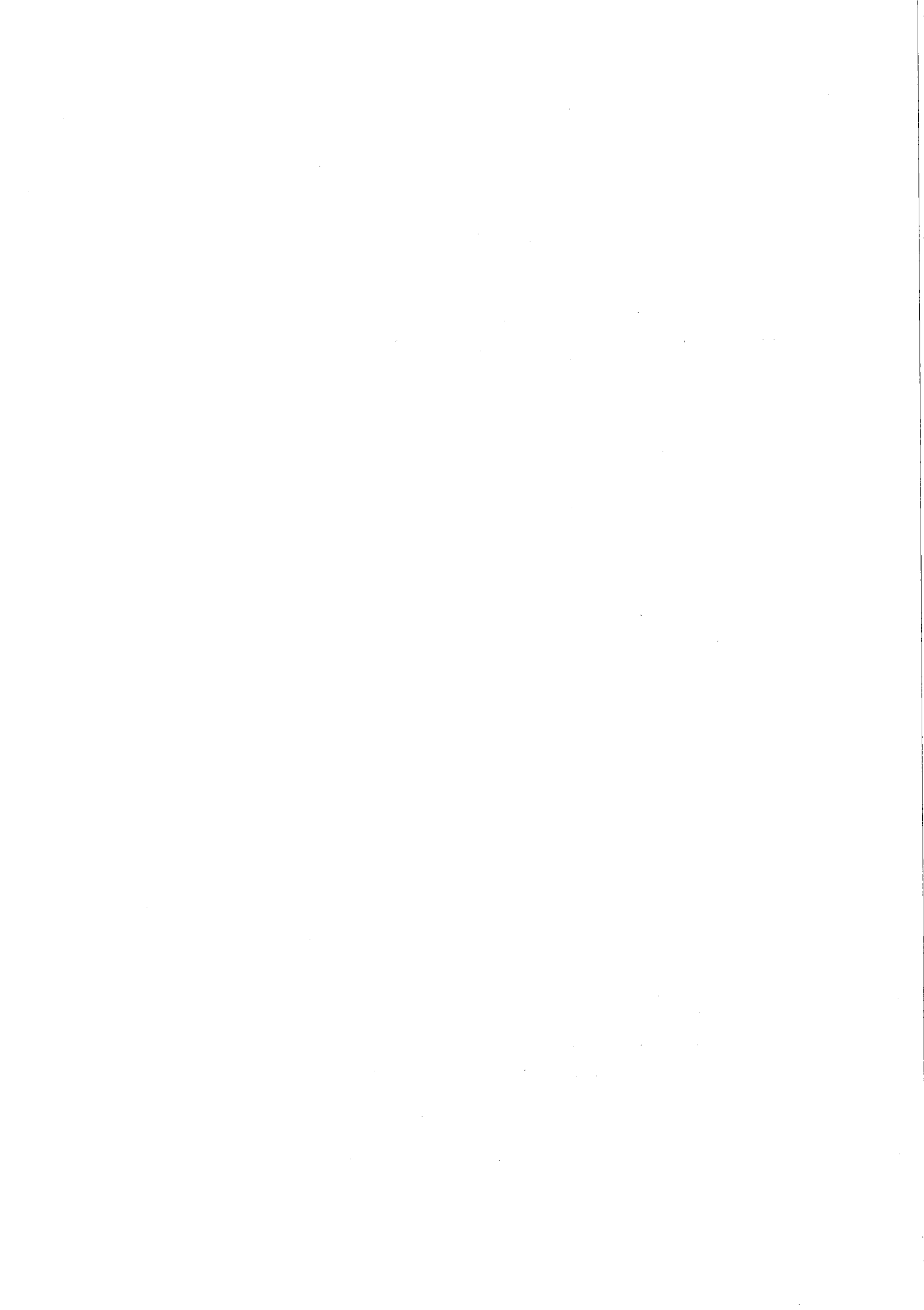
Introduction[1]

Within traditional game theory "the battle of the sexes" (BoS from now on) is the name of a game originating from Luce and Raiffa (1957). The game models a situation in which players wish to coordinate their behaviour, but have conflicting interests over outcome. A simple two person, two strategy version of the game is captured by the following normal form representation.

		2 I	2 II
1	I	2,1	0,0
	II	0,0	1,2

Given that player 1 chooses strategy I the best player 2 can do is also to play strategy I and vice versa. The same is true for strategy II. The strategy combinations (I,I) and (II,II) turn out to be (strict) Nash equilibria for this strategic game[2].

To find the equilibrium and to describe its characteristics in different settings, is in the centre of economic analysis. Within game theory, when modelling situations of strategic behaviour, the concept of Nash equilibrium is seen as the one giving most insight (van Damme; 1993).

There are growing awareness however, that the relevance of equilibrium analysis is not a straightforward one. The rationality assumption underlying traditional game theoretic equilibrium concepts are under attach for being too unrealistic and restrictive. Questions like "why is the use of Nash equilibrium appropriate" and "when will such an equilibrium analysis be suitable" are being raised and not easily answered. Traditional non-cooperative game theory has also little to offer in situations where multiple equilibria exist. In the example above, given the opportunity to choose, player 1 would prefer equilibrium (I,I) to manifest itself, and player 2 equilibrium (II,II). But with regard to the possibility of choosing among multiple equilibria, and eventually which one to choose, traditional game theory is silent. As an example of a BoS game, consider the formation of households by men and women. In forming households wife and husband have a desire to co-operate, but the output obtained by the two parts may differ according to the type of household (eg. patriarchal, egalitarian) established[3].

Evolutionary game theory is a theoretical perspective inquiring whether players will coordinate their behaviour, if they will play Nash and if so, which Nash equilibrium will

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[1] The paper has benefited from the comments of Michihiro Kandori, Jørgen Weibull, Ugurhan Berkok and Sjur Flåm.

[2] For a definition of a Nash equilibrium see. e.g. Osborne and Rubinstein (1994), p. 14.

[3] See e.g. Sen (1984) for a discussion of unequal distribution of goods within households.

be the outcome. This being the case, will evolutionary game theory be able to predict which Nash equilibrium, if any, a society of agents playing the BoS game will end up in? The paper proceeds as follows. First an outline of the different approaches within evolutionary game theory is given[4]. Thereafter the insight gained by these different approaches will be related to the BoS game presented. The aim is to investigate how and to what degree the different evolutionary game theoretic approaches are able to answer the questions raised.

Evolutionary game theory, an outline.

Evolutionary game theory is originating, as the name evolutionary implies, from biology (Maynard Smith and Price; 1973). In its original formulation, the players have no conscious choice with regard to which strategy to play, but are programmed to play certain strategies. These strategies may be modes of behaviour inherited from their forbearers or assigned to them by mutation. Output is defined as reproductive capacity or fitness.

The static approach.

The original formulation of the theory is a *static approach* which tries to capture a stable outcome of an evolutionary process. Members of a single population N are randomly drawn to interact with each other pairwise. In each match each player uses an action drawn from the set of available actions X. An output or fitness function u measures each players ability to survive, usually assessed as its number of surviving offsprings. If the player uses action x when he faces the distribution d of actions of its potential opponents, then its ability to survive is measured by the expectation of $u(x,z)$ under d, where x is the action chosen by the player and z the possible actions chosen by its opponents. This description corresponds to a two-player symmetric strategic game[5].

The concept of an evolutionary equilibrium is designed to capture the notion of a steady state in which no mutant can invade the population. For every possible action $x \in X$ the evolutionary process occasionally transforms a small fraction of the population into mutants who follow y. In an equilibrium any such mutant must obtain an expected payoff lower than that of the equilibrium action, so that the mutant dies out. If a fraction $\varepsilon > 0$ of the population are mutants using the action y while all the other players use action x, then

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[4] For more comprehensive and techically sophisticated overviews, see Van Damme (1993), Mailath (1993), Banerje and Weibull (1992), Hammerstein and Selten (1993) and Binmore and Samuelson (1993).

[5] A symmetric game has the following properties;

1) the number of pure strategies for each player is the same.

$$s_1 = s_2 = K = \{1,2,...k\} \quad \Delta = \left\{ x \in R_+^{\,k} : \sum x_i = 1 \right\}$$

2) the players position does not matter. $U_1(x,y) = U_2(y,x)$

the average payoff to a non-mutant must exceed the average payoff to a mutant for all values of ε sufficiently small;

$$u\big(x,(1-\varepsilon)x+\varepsilon y\big) > u\big(y,(1-\varepsilon)x+\varepsilon y\big)\ [6]$$

An equilibrium strategy satisfying this condition, is denoted an evolutionarily stable strategy (ESS).

Evolutionary game theory highlights that even in situations presuming no conscious choice or rationality, there may be a case for equilibrium analysis. Further, by employing the concept of ESS it may be possible to choose among multiple Nash equilibria. In a symmetric game, only symmetric Nash equilibria are candidates for ESS[7]. And among the symmetric Nash equilibria, only the ones based on strategies satisfying the evolutionary stability condition are ESS.

In non-symmetric games however, ESS is of less use with respect to refinement of Nash equilibria. In games with asymmetric player positions, only strict Nash equilibria can be ESS (Selten; 1980)[8]. Games without strict Nash equilibrium fail to have ESS. With regard to extensive form games, ESS is also of restricted value. An ESS has to reach all information sets in order to exclude alternative best responses (Selten; 1983).

The dynamic approach.

Within the dynamic approach the dynamics eventually leading to the equilibrium are of interest. The players, as they repeatedly interact, are assumed to try to maximise their stage-game payoff. In doing so, their choice of strategy in the next stage game is based on the belief that the distribution of their opponents play in the next game will be the same as the one revealed today. If they find it optimal, the players may choose to change strategy before the next stage game. In the general case, not all the players may adjust their behaviour in every period. Following Mailath (1993), given that the players, when they adjust, adjust towards the best reply of last periods distribution, this periods fraction of the population (p_t) playing a given strategy x will be given by the function;

$$p_t = b\big(p_{t-1}\big)$$

where $\big(p_{t-1}\big)$ is the fraction of the population playing strategy x last period.

$b(.)$ is referred to as the *selection dynamics*. By letting the selection dynamics satisfy the Darwinian property (that is; $b(p) < p$ when the expected output of playing strategy x is

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[6] With probability (1-ε) a player encounter a non-mutant, and with probability ε he encounters a mutant.

[7] A symmetric Nash equilibrium is a Nash equilibrium where the strategy x is the best reply against itself.

[8] If z is the best response to strategy x, and no other strategy is the best response to x, then the equilibrium (x,z) is a strict Nash equilibrium.

less than the expected output of an alternative strategy, and $b(p) \geq p$ otherwise), and adding the conditions that $b(0) = 0$, $b(1) = 1$ and the convention that $b(p) = p$ when p is such that the players are indifferent between which strategy to play (no adjustment in this case), we get a first order difference equation describing the evolution of the fraction of the population playing strategy x[9]. The important feature of any selection dynamics is that it always adjust behaviour in the population towards the current best reply. The population dynamics will always, if possible, increase the fraction playing the best reply in the population.

The *replicator dynamics* (Taylor and Jonker; 1978) originating from evolutionary biology is a particular, continuous time, selection mechanism satisfying the criterion of always adjusting population behaviour towards the current best reply. The dynamics for the population shares p_i is given by;

$$\dot{p}_i = \left[u\left(x^i, p\right) - u(p, p)\right]p_i$$

where $u(x^i, p)$ is the expected payoff to any pure strategy i at a random match when the population is in state p, and $u(p,p)$ is the average payoff to an individual in the population when the population is in state p[10]. The rate of change $\dfrac{\dot{p}_i}{p_i}$ of the population share using strategy i equals the difference between the strategy's current payoff, and the current average payoff in the population.

Within a biological context, the individual players are not assumed to adjust their behaviour. According to Weibull (1994), the replicators can be seen as the pure strategies within the game. These strategies can be copied without error from parent to child. In such a setting the individuals in the interacting population are merely the hosts of the replicators, and the replicator dynamics models how replicators compete for hosts in a population of pairwise interacting individuals. The replicator(s) resulting in the greatest biological fitness or reproductive success will be the winners. The population share programmed to a pure best reply to the current population state will have the highest growth rate, and the sub-populations associated with better-than-average strategies will grow while strategies associated with worse-than-average strategies will decline.

Compared to the static approach, dynamic evolutionary game theory shows not only that there may be a case for equilibrium analysis in economic analysis, but also how such a stable situation may be the outcome of a dynamic process. Within the replicator dynamics, Lyapunov stability derived without any rationality assumption, implies

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[9] Applied to the game given in the introduction, with regards to strategy I played by group 1, b(p) < p if p < 1/3, b(p) > p if p > 1/3 and b(1/3) = 1/3.

[10] replicator dynamics in a multi-population setting is:

$$\dot{p}_{ih} = \left[u\left(x_i^h, p_{-i}\right) - u_i(p)\right]p_i$$

The growth rate for subpopulation h using strategy i.

aggregate behaviour which appear rational and coordinated in the sense of a symmetric Nash equilibrium.

As in the static approach, the concept of ESS may be used to make refinements among the existing Nash equilibria. An ESS must be an asymptotically stable rest point of these kind of dynamics. To be asymptotically stable is a stronger requirement than to be Lyapunov stable[11]. Not all rest points that are Lyapunov stable will meet the requirements of an asymptotically stable point. That is, not all Nash equilibria will be ESS[12].

However, the approach have serious limitations. The assumption that the players are programmed to play certain strategies may describe the behaviour of some simple organic organisms (like bacteria), but hardly the behaviour of human beings. By letting the players be capable of adjusting their behaviour, the players get more human-like behaviour. But this adjustment process is normally based on imitation of other players strategies, either randomly as in pure imitation or deliberately as in payoff dependent imitation. Explicit modelling of human learning processes are usually lacking, and behaviour formation based on factors like role identification are not taken into consideration.

Evolutionary game theory reviewed thus far only tests the stability of the system against single mutants. Mutants are assumed to be relatively rare, only one mutant comes into being at a time. The system will thus have time to settle back into the original position before the next mutant comes about. By assuming uniform random mixing of the pairwise players, the probability of meeting some specific type does not depend on your own type. In a situation where mutants would mainly interact with each other, they might enter the system more easily.

As in the static approach the usefulness of ESS is restricted outside the realm of one population symmetric games. For asymmetric games the same critique as for the static approach applies, and within a multi-population setting there is no clear definition of ESS (Weibull; 1994).

Neither the static nor the dynamic approach provides any solution to the problem of how to choose among strict Nash equilibria. Within the replicator dynamics any strict Nash equilibrium is an asymptotically stable stationary point of the dynamic (Weibull; 1993). Hence, the evolutionary process does not help in selecting among such equilibria. The theory cannot explain why beliefs are coordinated on a specific equilibrium. Having this coordination of beliefs as the point of departure, the equilibrium chosen will depend on this departure point. The resulting equilibrium will be path and history dependent.

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[11] For a discussion of Lyapunov and asymptotically stable rest points see Weibull (1994).

[12] Within the replicator dynamics, a Nash equilibrium is a stationary point of the dynamic. Each stable stationary point is a Nash equilibrium, and an asymptotically stable fixed point is a perfect equilibrium (Bomze; 1986). If we allow for mixed strategies to be inherited, then asymptotically stable fixed points of the replicator dynamics correspond exactly to ESS (Bomze and Van Damme; 1982, Hines; 1980, Zeeman; 1981). If only pure strategies can be inherited, being an ESS is sufficient but not necessary for asymptotic stability (Taylor and Jonker; 1978).

The stochastic approach.

Foster and Young (1990) and later Fudenberg and Harris (1992) and Kandori, Mailath and Rob (1993) showed that it may be possible to discriminate between strict Nash equilibria by adding perpetual randomness (or statistical noise) to the system. The models differ in their assumptions regarding time (continuous/discrete), population size (finite/infinite) and dynamic formulation. We will base the presentation of the stochastic approach on the model of Kandori, Mailath and Rob (1993) (from now on KMR) who assume discrete time and finite population size.

The game is played by the members of a finite population of size N. In each period the members of the population are randomly matched into pairs. Given two strategies, the state of the system may be characterized by z_t, the number of agents playing a specific strategy in a given time period. The set of possible values of z_t is $Z = \{0,1,....,N\}$.

The players are capable of changing their strategies, but the opportunity to adjust the strategy choice is assumed to arrive stochastically and independently across players and time. When the opportunity is there, the players are assumed to choose a myopic best response, and a law of motion towards myopic best responses is established. This law of motion, which satisfies the Darwinian properties, is called *the best response dynamic*. In a 2x2 game, the dynamic is given by the rule:

$$B(z) = \begin{cases} N & \text{if } \pi_1(z) > \pi_2(z) \\ z & \text{if } \pi_1(z) = \pi_2(z) \\ 0 & \text{if } \pi_1(z) < \pi_2(z) \end{cases}$$

where $\pi_1(z)$ is the expected payoff by playing strategy I when the system is in state z, and $\pi_2(z)$ the expected output by playing strategy II.

In addition to the best response dynamics, noise is added to the system. The noise comes about by assuming that a player who is expected to play strategy x by the best response dynamic, with a small but positive probability ε mutates and plays strategy y[13]. The probability to mutate is assumed independent across players and over time.

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[13] In a setting with more than two strategy choices, the probability that a player expected to play strategy k mutates to strategy j is $m_j\varepsilon > 0$, where ε is the probability that the player will mutate instead of playing according to the best response dynamic and m_j the probability that he will mutate to strategy j. $\Sigma_j m_j=1$ and $\varepsilon, m_{,j} \in (0,1)$

The evolution of the system is now described by a Markov chain[14] which is finite, irreducible and aperiodic[15]. Given such a Markov chain all states have a strictly positive probability of being observed in the limit as time tends to infinity.

In a situation without noise, the strict Nash equilibria of the game will be absorbing states of the best reply dynamics. When noise is added, all states may be realized. All these states however, will not have the same likelihood of being observed. A Markov chain which is aperiodic and irreducible has a unique stationary distribution. This stationary distribution is stabile and ergodic. Stability implies that, starting at any initial distribution over states, the unconditional distribution of states converges to the stationary distribution asymptotically. Ergodicity on the other hand, means that the stationary distribution asymptotically describes the time average behaviour of the process. Given the properties of stability and ergodicity, when ε is small, the probability of observing the system being in one of the absorbing states is much bigger than the probability of observing it in any other state. This being the case, the analysis of players coordinating their behaviour on a particular equilibrium, may be confined to the set of stationary states only.

The goal of the stochastic analysis is to characterize the limit (stationary) distribution, given that it exists, when ε goes to zero. States in the support of the limit distribution are denoted long run equilibrium (KMR) or stochastically stable states (Foster and Young; 1990).

Long run equilibrium (denoted LRE) is a stochastic equilibrium concept. This implies that the system, once settled down, will not stay in the long run equilibrium for ever. The mutants make the system move from one equilibrium to another. The average time spent in the different equilibria however, will differ, with the longest average time spent at the long run equilibrium. The probability of observing the system in the long run equilibrium will, as a result, be greater than the probability of observing the system in any other state.

The long run equilibrium.

For general 2x2 symmetric games with two strict symmetric Nash equilibria, KMR showed that the limit distribution places probability one on the state in which all players play the strategy with the larger basin of attraction under the best reply dynamic. In other words, the equilibrium state with the biggest basin of attraction is the long run equilibrium.

To exemplify this result, consider the following 2x2 symmetric game played by 10 players;

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[14] Within a Markov chain today's distribution of states depends only on yesterdays distribution.
[15] A Markov chain is irreducible if Prob(z(T)=z' | z(0)=z) > 0 for all z and z' for some T $\in$ N (N is the set of natural numbers). That is; any pair of states within the system are mutually reachable. It is aperiodic if the greatest common divisor of {T $\in$ N | Prob(z(T)=z'| z(0)=z) > 0 } is 1 for all z and z'. There is no cycle in the dynamic.

	2	
	I	II
1 I	2,2	0,0
II	0,0	1,1

The possible states z_t of this system (the number of agents playing strategy I) is 0,1,...,.10, and the steady states are 0 or 10. If $\frac{10}{3}$ (the "mixed" strategy equilibrium) had been an integer, $\frac{10}{3}$ would been the third steady state of the dynamic.

To find the basin of attraction belonging to the two equilibria, the critical level z^* is computed. z^* is the level for which the following is true (z^* needs not be an integer):

$$\text{sign } (\pi_I(z) - \pi_{II}(z)) = \text{sign } (z - z^*)$$

For general symmetric games with payoff matrix:

	2	
	I	II
1 I	a,a	b,c
II	c,b	d,d

KMR (1993) showed that $z^* = \dfrac{N(d-b)+a-d}{a-c+d-b}$. Essentially, z^* corresponds to the mixed strategy equilibrium which put probability $\mu = \dfrac{d-b}{a-c+d-b}$ on strategy I. z^* is not exactly equal to μN however, since players with strategies I and II face slightly different strategy distributions due to the finiteness of the population. But as the population size becomes larger, the difference between z^* and μN vanishes.

The two states 0 and 10 have basins of attraction under b given by $\{z < z^*\}$ and $\{z > z^*\}$ respectively. The critical value z^* divides the state space between 3 and 4. The basin of attraction for equilibrium 10 consist of the states 4,..,10, and the basin of attraction for equilibrium 0 of the states 0,..,3. Given that the system is in the equilibrium 0 (all play strategy II), at least 4 mutations are needed to reach the basin of attraction of equilibrium 10. When starting in equilibrium 10 on the other hand, the minimum of 7 mutations are required to reach the other equilibrium[16]. It takes more mutations to upset equilibrium 10 than to upset equilibrium 0, resulting in 10 being the long run equilibrium.

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[16] In the general case, assuming $z^* = \mu N$, to escape the basin of attraction of equilibrium (d,d) and reach the basin of attraction of equilibrium (a,a), a number of dN/(a+d) mutations are needed. To go from equilibrium (a,a) to (d,d) on the other hand, N-[dN/(a+d)]=aN/(a+d) mutations are required.

Given that the system is at one of the absorbing states, for the system to reach the basin of attraction of the other absorbing state, it is crucial that enough mutations occur in one period. Even though there are other sequences of events that will take the system from one absorbing state into the basin of attraction of the other, the lowest order probability event is the one in which the transition occurs in one period. To see this, assume that the mutations instead are spread over two periods. After the first mutation period and before the second, the system is still in the basin of the original equilibrium. This being the case, at least one of the mutant players will switch back to the original strategy by the best response dynamic, and this effect must be overcome by one more mutation.

Theorem 3 in KMR (1993;44) states that: *given that the stage game is a coordination game and $z^* \neq N/2$, for any population size $N \geq 2$ and any adjustment process satisfying the Darwinian properties, the limit distribution puts probability 1 on N if $z^* < N/2$ and on 0 if $z^* > N/2$.*

In our example $z^* = \frac{10(1-0)+2-1}{2-0+1-0} = \frac{11}{3} = 3\frac{2}{3}$. Since $N = 10$ and $z^* < N/2$, the basin of attraction supporting equilibrium N is bigger than the basin of attraction supporting equilibrium 0. All players choosing strategy I is the long run equilibrium in this game.

Evolutionary game theory and the battle of the sexes

In this section we will return to the questions raised in the introduction regarding the ability of evolutionary game theory to predict which, if any, Nash equilibrium a BoS game, e.g. a game of men and women household builders, will end up in.

The game "the battle of the sexes" (BoS) put forward in the introduction, is an asymmetric, normal form game. The game has two symmetric strict Nash equilibria in pure strategies, (I,I) and (II,II), and a non-symmetric equilibrium in mixed strategies $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$. In this asymmetric game, the two strict Nash equilibria are candidates for ESS. By the use of a selection dynamics satisfying the Darwinian property, the law of motion in the game is given by figure 1. The mixed strategy equilibrium turns out to be unstable, while the two strict Nash equilibria are asymptotically stable states of the system. In equilibrium $A = (NxN)$ all players in both populations chooses to play strategy I. In $A' = (0x0)$ on the other hand, all of them chooses to play strategy II.
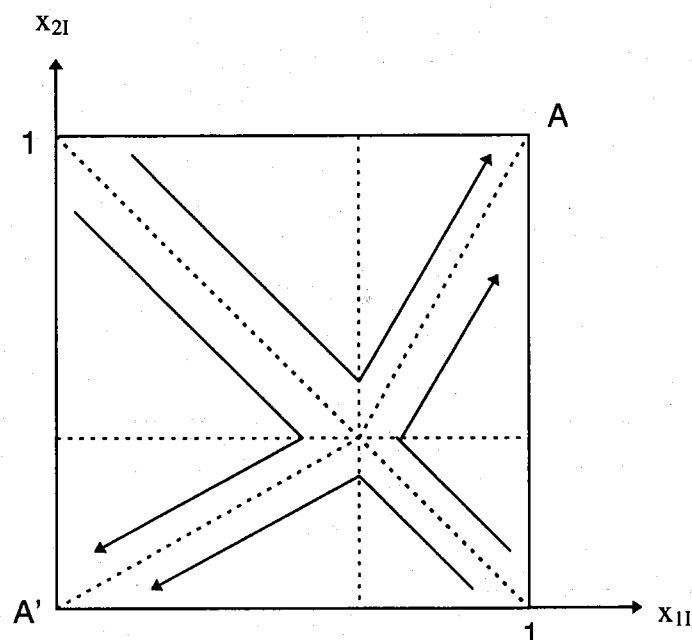
<u>Figure 1.</u>

With respect to the game in question, *the dynamic evolutionary approach* predicts the society to end up in either A or A'. But which one, the theory cannot tell. To be able to determine the most likely candidate, one has to move beyond the dynamics of the model and into the history of the game.

In 2x2 symmetric, singe population games, *the stochastic evolutionary approach* formulated by KMR is capable of discriminating between different stationary states on the basis of the number of states in support of each state[17]. In the following we will make use of their results in our inquiry to find the long run equilibrium for the BoS game presented.

The BoS game may be said to differ from the simple coordination game of the previous section in two respects. First by being asymmetric rather than symmetric, and related to this, by being played by players from two, rather than one, population. To capture these differences, the technique employed to find the basins of attraction corresponding to the different stationary states in the 2x2 symmetric, one population game, must be modified to capture the characteristics of the game in question.

The modifications needed will be presented in two steps. We start with the ones required to go from a one population to a two population 2x2 symmetric game. Thereafter the changes making the technique suitable for asymmetric 2x2 games will be put forward.

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[17] The cost of transition is measured by the number of mutations required to go from one equilibrium to the basin of attraction of another in one leap. As N increases, the number of mutations needed also increases, making a transition less probable. This feature makes the analysis most suitable for relatively small populations. By introducing the feature of local interaction among the players, the analysis is made independent of N (see.Ellison ; 1993)

Two populations, 2x2 symmetric game.

Imagine a 2x2 symmetric game where the players playing against each other come from two different populations, each containing N players. The state space for this game becomes $Z = (0,1,...,N)x(0,1,...,N)$. The state space is a grid of points over a two dimensional square. The possible states at each time is now described by a pair of numbers; $z = (z_{1i}, z_{2i})$. The state indicates the number of players in each population playing strategy i (at time t).

For each population separately, the Darwinian adjustment dynamics specified for the one population setting, is assumed. Furthermore, since a player from population 1 is sure to meet a player from population 2 within each stage game, players belonging to population 1 are only interested in the distribution of strategies played by players of population 2, and vice versa. This being the case, a critical level z_1*, z_2* for each population can be computed. The state space is now partitioned into four subregions. In subregions B and B' the dynamics are unambiguous. They point towards one of the game's two Nash equilibria. In subregions C and D on the other hand, the Darwinian dynamics point in conflicting directions. The state space and the critical levels for a symmetric, two population 2x2 game is given in figure 2. The arrows indicate the Darwinian dynamics.
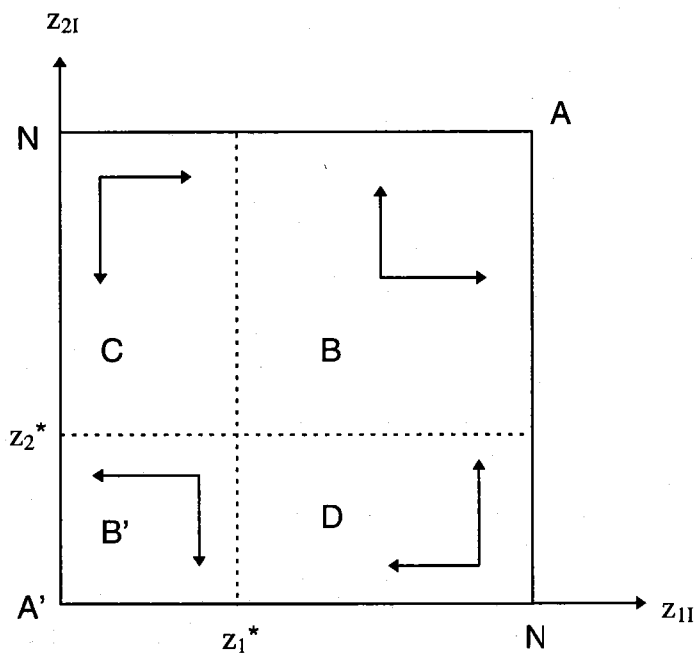

Figure 2.

To find the long run equilibrium, the cost of transition between the two equilibria A and A' must be computed. The cost of transition between the two equilibria is equal to the minimial number of mutations needed to move from one equilibrium into the basin of attraction of the other. Under the assumption that the adjustment dynamics within the two

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populations of players are identical, (implying that the speed of adjustment of population 1 players are equal to speed of adjustment of population 2 players), the basin of attraction for A and A' are given by areas B and B' (the areas with unambiguous dynamics) respectively.

The cost of transition from A' to A, $C(A',A) = z_i*$[18], where $i = 1,2$[19]. To see this, suppose that all players initially choose strategy II (equilibrium A'). If sufficient number of population 1 players mutate towards strategy I, strategy I becomes the best reply for population 2 players the next period. By the assumtion of stochastic adjustment, it happens with positive probability that, in the next period, none of the players from population 1 adjust but all players from population 2 switch to strategy I. In this way, equilibrium A, where I is played can be achieved by mutations in one of the populations only. By the same line of reasoning, the cost of transition from A to A', $C(A,A') = N-z_i*$. In our example, z_i* is smaller than $N-z_i*$, and A becomes the long run equilibrium.

To illustrate the effect of variable adjustment speed, consider the case where population 1 players adjust infinitely faster than population 2 players in region D, and population 2 players adjust infinitely faster than population 1 players in region C. In this case the basin of attraction for equilibrium A' consists of region B',C and D, while the basin of attraction for equilibrium A consist of B alone. In this situation, to escape the A equilibrium, $N-z_i*$ mutations are needed. To escape the A' equilibrium, region B must be reached. This requires $2(z_i*)$ mutations (z_i* mutations in each population). To find the long run equilibrium, $N-z_i*$ must be compared to $2(z_i*)$. In the example presented, $N-z_i*$ turns out to be smaller than $2(z_i*)$, resulting in A' being the long run equilibrium.

Two populations, 2x2 asymmetric game.

In the asymmetric BoS game, z_1* and z_2* can be computed in the same way as in the symmetric two population 2x2 game. This time however, z_1* is not identical to z_2*. The critical levels, and the subregions of the state space exhibiting unambigious and conflicting dynamics, is shown in figure 3. As before, given identical adjustment speed within the two populations, the areas B and B' exhibiting unambigous adjustment dynamics are the basins of attraction supporting equilibrium A and A'.

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[18] This is an approximation. Since we are dealing with finite populations, the cost of transition is equal to the integer just exeeding z_i*. The same applies to $N-z_i*$.
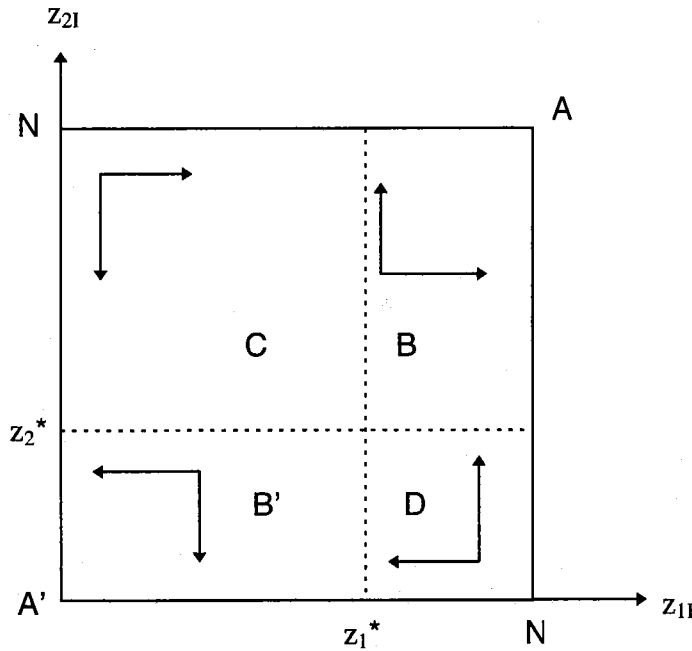[19] In a symmetric two population game, z_1* equals z_2*.

Figure 3.

By the same line of reasoning as in the symmetric, two population case, the cost of transition between the two equilibria are equal to;

$$C(A',A) = \min(z_1{}^*, z_2{}^*) \text{ and } C(A',A) = \min(N - z_1{}^*, N - z_2{}^*).$$

If $\min(z_1{}^*, z_2{}^*) < \min(N - z_1{}^*, N - z_2{}^*)$, A will be LRE and if $\min(z_1{}^*, z_2{}^*) > \min(N - z_1{}^*, N - z_2{}^*)$ A' is the long run equilibrium. In the example given, $\min(z_1{}^*, z_2{}^*)$ turn out to be equal to $\min(N - z_1{}^*, N - z_2{}^*)$. The cost of transition from A to A' is equal to the cost of transition from A' to A. This being the case, both the equilibria are LRE. The system will as a result cycle between them, spending half the time in each.

For general 2x2 symmetric games, KMR showed that the limit distribution put probability one on the equilibrium with the biggest basin of attraction. We state that this result is applicable also for 2x2 asymmetric games. For the BoS game in question, the basin of attraction corresponding to equilibrium A' is of size $(z_1{}^*)(z_2{}^*)$, and the basin supporting A is equal to $(N - z_1{}^*)(N - z_2{}^*)$. According to the size criteria, A is LRE if $(z_1{}^*)(z_2{}^*) < (N - z_1{}^*)(N - z_2{}^*)$ and A' is LRE if the converse holds.

In confirming this statement we will only consider the case of $(z_1{}^*)(z_2{}^*) < (N - z_1{}^*)(N - z_2{}^*)$ since the case $>$ is just a re-labelling of the $<$ situation. We will show that when $(z_1{}^*)(z_2{}^*) < (N - z_1{}^*)(N - z_2{}^*)$, the cost of transition in going from A' to A is smaller than the cost of going from A to A'. Recall that the costs of

transition are given by; $C(A',A) = \min(z_1{}^*, z_2{}^*)$ and $C(A,A') = \min(N - z_1{}^*, N - z_2{}^*)$. For a transition to occur it is sufficient that enough mutations occur within one of the populations.

Proposition:

$$\min(z_1, z_{-1}) < \min(N - z_1)(N - z_{-1}) \quad \Leftrightarrow \quad z_1 + z_{-1} < N \quad \Leftrightarrow \quad z_1 z_{-1} < (N - z_1)(N - z_{-1})$$

Proof:

Observe that: $\quad z_i = \min(z_1, z_{-1}) \quad \Leftrightarrow \quad N - z_{-i} = \min(N - z_1, N - z_{-1})$

Hence: $\quad z_i = \min(z_1, z_{-1}) \quad < \quad N - z_{-i} = \min(N - z_1, N - z_{-1})$

$$\Leftrightarrow \quad z_1 + z_{-1} < N \quad \Leftrightarrow \quad p_1 + p_{-1} < 1 \qquad \left(\text{where } p_i = \frac{z_i}{N} \right)$$

$$\Leftrightarrow \quad p_1 p_{-1} + p_1 + p_{-1} - p_1 p_{-1} < 1 \quad \Leftrightarrow \quad p_1 p_{-1} < 1 - p_1 - p_{-1} + p_1 p_{-1}$$

$$\Leftrightarrow \quad p_1 p_{-1} < (1 - p_1)(1 - p_{-1}) \quad \Leftrightarrow \quad z_1 z_{-1} < (N - z_1)(N - z_{-1})$$

The result shows that $(z_1{}^*)(z_2{}^*) < (N - z_1{}^*)(N - z_2{}^*)$ is equivalent to $\min(z_1{}^*, z_2{}^*) < \min(N - z_1{}^*, N - z_2{}^*)$. The equilibrium supported by the bigger basin of attraction is LRE. The result also shows that $(z_1{}^*)(z_2{}^*) < (N - z_1{}^*)(N - z_2{}^*)$ is equivalent to $z_1{}^* + z_2{}^* < N$. The criteria is a generalisation of the result obtained by KMR for 2x2 symmetric games. In symmetric games, $z_1{}^* = z_2{}^*$ and the criteria becomes $z^* < N/2$.

The BoS game given in the introduction is illustrated in figure 4. For N = 10, $z_1{}^* + z_2{}^* = 10 = N$. The basin of attraction corresponding to each equilibrium is of equal size. Both the equilibria is LRE. The result is, as expected, equal to the result obtained by comparing $C(A',A)$ and $C(A,A')$.

The result presented shows that when $z_1{}^* + z_2{}^* < N$, equilibrium A characterised by all the players choosing strategy I will be long run equilibrium. Generally, however, the sign of $z_1{}^* + z_2{}^* - N$ is a function of the risk characteristics of the BoS stage game. The risk characteristics is the risk connected with strategy I relative to strategy II. If c_i is large relative to b_i, strategy I is riskier than strategy II. Harsanyi and Selten (1988) proposed a notion of risk dominance to capture the risk characteristics of a 2x2 game; Their notion of risk dominates states that: *Equilibrium A (all players play strategy I) risk dominates equilibrium A' (all*

players play strategy II) if $(a_1 - c_1)(a_2 - c_2) > (d_1 - b_1)(d_2 - b_2)$ *and equilibrium A' risk dominates A if* $(a_1 - c_1)(a_2 - c_2) < (d_1 - b_1)(d_2 - b_2)$.

To obtain the risk characteristics of the BoS stage game, consider the game written in general form;

<div align="center">

2

		I	II
1	**I**	a_1, a_2	b_1, c_2
	II	c_1, b_2	d_1, d_2

</div>

By making use of the fact that $z_i^* = \dfrac{N(d_i - b_i) + a_i - d_i}{a_i - c_i + d_i - b_i}$, where $i = 1,2$, it can be shown that $z_1^* + z_2^* \leq N$ if and only if:

$$\left(1 - \frac{1}{N}\right)(a_1 - c_1)(a_2 - c_2) - \frac{1}{N}(a_1 - b_1)(a_2 - b_2) \geq \left(1 - \frac{1}{N}\right)(d_1 - b_1)(d_2 - b_2) - \frac{1}{N}(d_1 - c_1)(d_2 - c_2)$$

As N increases, this condition approaches the Harsanyi & Selten notion of risk dominance.

For N big enough, that is for

$$N \geq 1 + \frac{(a_1 - b_1)(a_2 - b_2) - (d_1 - c_1)(d_2 - c_2)}{(a_1 - c_1)(a_2 - c_2) - (d_1 - b_1)(d_2 - b_2)}$$

the long run equilibrium chosen will be the risk dominant equilibrium.

For games with *player-specific* risks, that is for games where the players face identical risk regardless of the strategy chosen, but where players from the different populations face different risk, $b_i = c_i$, ($i = 1,2$), the condition for $z_1^* + z_2^* \leq N$ reduces to;

$$(a_1 - c_1)(a_2 - c_2) > (d_1 - c_1)(d_2 - c_2)$$

Since c_i is the lowest output attainable, $(k_i - c_i)$, where $k = a, d$ and $i = 1,2$, is the surplus obtained given an equilibrium outcome. In a game with player-specific risk the long run equilibrium will be the equilibrium with the largest surplus product. Note that for such games the population size N do not influence the equilibrium chosen.

Should the risk be identical for the two groups of players, $b_i = c_i = c$, differences in equilibrium outputs determines the long run equilibrium. This is most easily seen by choosing $c = 0$. The condition then reduces to $a_1 a_2 > d_1 d_2$. The equilibrium with the largest product of outputs is the long run equilibrium.

Concluding Remarks

By extending the stochastic evolutionary model of KMR to cover the asymmetric 2x2 BoS game, the long run equilibrium of the game is selected. For population sizes sufficiently large the long run equilibrium is the risk dominant equilibrium. With respect to the specific game presented in the introduction, both the strict Nash equilibria turned out to be LRE, indicating that the system will cycle between them.

Within the model formulation presented, the size of the basin of attraction is the important feature of the selection dynamic. This feature however, is peculiar to discrete models with independent mutations having equal probabilities to occur. The result may not carry over to other model formulations.

Interesting aspects not captured by the model presented are:
- given differences in the mutation rates in different populations, which relationship, if any, can be found between a fast/slow mutation rate and the characteristics of the long run equilibrium chosen.
- is the mutation rate exogenously given? What would happen if the tendency to mutate depends on the number of mutant players within the players own population group (the number of role models).
- will the sizes of the interacting populations influence the equilibria chosen? In a setting where the individuals are randomly drawn to interact pair-wise, unequal population sizes must imply that not all individuals in the largest population gets to play in each stage game. Will the distribution of strategies within the subgroup playing be representative for the group as a whole, or is there some selection mechanism making the distribution skewed toward a/some specific strategies?
- A game where the players only learn about the distribution of strategies played in the opposite population, can model a situation where the game is assumed to take place only once in each generation. To catch the dynamics of a repeated game played by the same two players in every period on the other hand, some learning process regarding the opponents strategy choices must be included.
- will the assumption of players randomly mixed in pairs catch the team-formation procedure in real life situations?

The list reveals many areas for research where the method of evolutionary game theory can be applied and developed. Being able to predict not only the equilibrium states possible, but also the one most probably chosen in different game theoretic settings, is an existing challenge increasing the usefulness of game theoretic analysis.

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